

ANALOGUES OF AUSLANDER–YORKE THEOREMS FOR MULTI-SENSITIVITY

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ABSTRACT. We study multi-sensitivity and thick sensitivity for continuous surjective selfmaps on compact metric spaces. Our main result states that a minimal system is either multi-sensitive or an almost one-to-one extension of its maximal equicontinuous factor. This is an analog of the Auslander-Yorke dichotomy theorem: a minimal system is either sensitive or equicontinuous. Furthermore, we introduce the concept of a syndetically equicontinuous point, and prove that a transitive system is either thickly sensitive or contains syndetically equicontinuous points, which is a refinement of another well known result of Akin, Auslander and Berg.

1. INTRODUCTION

Throughout this paper (X, T) denotes a *topological dynamical system*, where X is a compact metric space with metric ϱ and $T : X \rightarrow X$ is a continuous surjection.

The notion of sensitivity (sensitive dependence on initial conditions) was first used by Ruelle [23]. According to the works by Guckenheimer [15], Auslander and Yorke [6] a dynamical system (X, T) is called *sensitive* if there exists $\delta > 0$ such that for every $x \in X$ and every neighborhood U_x of x , there exist $y \in U_x$ and $n \in \mathbb{N}$ with $\varrho(T^n(x), T^n(y)) > \delta$, where \mathbb{N} is the set of all natural numbers (positive integers). According to [4], it is easy to see that (X, T) is sensitive if and only if $S_T(U, \delta)$ is infinite for some $\delta > 0$ and every open¹ set $U \subset X$, where

$$S_T(U, \delta) = \{n \in \mathbb{N} : \text{there are } x_1, x_2 \in U \text{ such that } \varrho(T^n x_1, T^n x_2) > \delta\}.$$

We define $J_T(U, \delta) \subset \mathbb{N}$ to be the complement of $S_T(U, \delta)$.

The Lyapunov stability or, in other words, equicontinuity is the opposite to the notion of sensitivity. Recall that a point $x \in X$ is called *Lyapunov stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varrho(x, x') < \delta$ implies $\varrho(T^n x, T^n x') < \varepsilon$ for any $n \in \mathbb{N}$, equivalently, for every $\varepsilon > 0$ there exists a neighborhood U of x such that $J_T(U, \varepsilon) = \mathbb{N}$. This condition says exactly that the sequence of iterates $\{T^n : n \geq 0\}$ is equicontinuous at x . Hence, such a point is also called an *equicontinuity point* of (X, T) . Denote by $\text{Eq}(X, T)$ the set of all equicontinuity points of (X, T) . The system (X, T) is called *equicontinuous* if $\text{Eq}(X, T) = X$.

The well-known Auslander-Yorke dichotomy theorem states that a minimal dynamical system is either sensitive or equicontinuous [6] (see also [14]), which was further refined in [1, 2]: a transitive system is either sensitive or almost equicontinuous (in the sense of containing some equicontinuity points). We recommend [20]

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¹Because we so often have to refer to open, nonempty subsets, we will call such subsets *opene*.

for a survey on the recent development of chaos theory, including sensitivity and equicontinuity, in topological dynamics.

Recall that a subset $\mathcal{S} \subset \mathbb{N}$ is called *thick* if for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $\{n_k, n_k + 1, \dots, n_k + k\} \subset \mathcal{S}$, and is *syndetic* if there exists $m \in \mathbb{N}$ such that $\mathcal{S} \cap \{n, n + 1, \dots, n + m\} \neq \emptyset$ for each $n \in \mathbb{N}$. A thick set has a nonempty intersection with every syndetic set. The following definitions of stronger forms of sensitivity were introduced in [21, 22]. A dynamical system (X, T) is said to be

- (1) *thickly sensitive* if there exists $\delta > 0$ such that $S_T(U, \delta)$ is thick for any opene $U \subset X$;
- (2) *multi-sensitive* if there exists $\delta > 0$ such that $\bigcap_{i=1}^k S_T(U_i, \delta) \neq \emptyset$ for any finite collection U_1, \dots, U_k of opene subsets of X .

In the paper we show that an analog of the Auslander-Yorke dichotomy theorem can also be found for this stronger forms of sensitivity. Precisely, by using Veech's characterization of equicontinuous structure relation of a system [25, Theorem 1.1], we prove that a minimal system is either thickly sensitive or an almost one-to-one extension of its maximal equicontinuous factor (Theorem 3.1), and thick sensitivity is equivalent to multi-sensitivity for transitive systems (Proposition 3.2). In particular, an invertible minimal system is either multi-sensitive or almost automorphic (Corollary 3.3).

Recall that the concept of almost automorphy, as a generalization of almost periodicity, was first introduced by Bochner in 1955 (in the context of differential geometry [8]) and studied by many authors starting from [9], [24], [26].

We extend the notion of equicontinuity by demanding that the set $J_T(U, \varepsilon)$ is large. Precisely, we introduce the concept of syndetically equicontinuous points, which means that for every $\varepsilon > 0$ there exists a neighborhood U of x such that $J_T(U, \varepsilon)$ is a syndetic set. It turns out that this new notion of local equicontinuity is very useful. That is, the refined Auslander-Yorke dichotomy theorem [1, 2] also holds in our setting (Theorem 3.4): a transitive system is either thickly sensitive or contains syndetically equicontinuous points.

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2. PRELIMINARIES

In this section we recall standard concepts and results used in later discussions.

2.1. Topological dynamics. Recall that (X, T) is (*topologically*) *transitive* if $N_T(U_1, U_2) = \{n \in \mathbb{N} : U_1 \cap T^{-n}U_2 \neq \emptyset\}$ is nonempty for any opene $U_1, U_2 \subset X$. A point $x \in X$ is called *transitive* if its *orbit* $\text{orb}_T(x) = \{T^n x : n = 0, 1, 2, \dots\}$ is

dense in X . Denote by $\text{Tran}(X, T)$ the set of all transitive points of (X, T) . Since T is surjective, (X, T) is transitive if and only if $\text{Tran}(X, T) \neq \emptyset$.

The system (X, T) is called *minimal* if $\text{Tran}(X, T) = X$. In general, a subset A of X is *invariant* if $TA = A$. If A is a closed, nonempty, invariant subset then $(A, T|_A)$ is called the associated *subsystem*. A *minimal subset* of X is a nonempty, closed, invariant subset such that the associated subsystem is minimal. Clearly, (X, T) is minimal if and only if it admits no a proper, nonempty, closed, invariant subset. A point $x \in X$ is called *minimal* if it lies in some minimal subset. Zorn's Lemma implies that every closed, nonempty invariant set contains a minimal set. Observe that by the classic result of Gottschalk, $x \in X$ is minimal if and only if $N_T(x, U) = \{n \in \mathbb{N} : T^n x \in U\}$ is syndetic for any neighborhood U of x .

A transitive system (X, T) is called an *E-system* [14] if it admits an invariant probability Borel measure μ with full support, that is $T\mu = \mu$ and $\mu(U) > 0$ for all open $U \subset X$. Note that a nonminimal E-system is sensitive [14, Theorem 1.3].

2.2. Extensions and factor maps. Let (X, T) and (Y, S) be topological dynamical systems. By a *factor map* $\pi : (X, T) \rightarrow (Y, S)$ we mean that $\pi : X \rightarrow Y$ is a continuous surjection with $\pi \circ T = S \circ \pi$. In this case, we call (X, T) an *extension* of (Y, S) and (Y, S) a *factor* of (X, T) , we also call $\pi : (X, T) \rightarrow (Y, S)$ an *extension*.

Each dynamical system admits a maximal equicontinuous factor. In fact, this factor is related to the regionally proximal relation of the system. The *regionally proximal relation* $Q_+(X, T)$ of (X, T) is defined as: $(x, y) \in Q_+(X, T)$ if and only if for any $\varepsilon > 0$ there exist $x', y' \in X$ and $n \in \mathbb{N}$ with $\max\{\varrho(x, x'), \varrho(y, y'), \varrho(T^n x', T^n y')\} < \varepsilon$. Observe that $Q_+(X, T) \subset X \times X$ is closed and positively invariant (in the sense that if $(x, y) \in Q_+(X, T)$ then $(Tx, Ty) \in Q_+(X, T)$), and the quotient by the smallest closed, positively invariant equivalence relation containing it is the maximal equicontinuous factor $(X_{\text{eq}}, T_{\text{eq}})$ of (X, T) . If (X, T) is minimal, then $Q_+(X, T)$ is in fact an equivalence relation by [5, 7, 11, 25] and [17, Proposition A.4]. Denote by $\pi_{\text{eq}} : (X, T) \rightarrow (X_{\text{eq}}, T_{\text{eq}})$ the corresponding factor map. Remark that $(X_{\text{eq}}, T_{\text{eq}})$ is invertible, when (X, T) is transitive, because each transitive equicontinuous system is uniformly rigid [14, Lemma 1.2] and this implies invertibility.

Let X be a compact metric space and let $\phi : X \rightarrow Y$ be a continuous surjective map. Denote by $Y_0 \subset Y$ the set of all points $y \in Y$ whose fibers are singletons. The set Y_0 is a G_δ subset of Y , because

$$Y_0 = \{y \in Y : \phi^{-1}(y) \text{ is a singleton}\} = \bigcap_{n \in \mathbb{N}} \left\{ y \in Y : \text{diam}(\phi^{-1}(y)) < \frac{1}{n} \right\}$$

and the map $y \mapsto \text{diam}(\phi^{-1}(y))$ is upper semi-continuous. Recall that the function $f : Y \rightarrow \mathbb{R}_+$ is *upper semi-continuous* if $\limsup_{y \rightarrow y_0} f(y) \leq f(y_0)$ for each $y_0 \in Y$.

If $Y_0 \subset Y$ is a dense subset, then we call ϕ *almost one-to-one*. If $\pi : (X, T) \rightarrow (Y, S)$ is an almost one-to-one factor map between topological dynamical systems, then we also call (X, T) an *almost one-to-one extension* of (Y, S) . Recall that if a dynamical system (X, T) is minimal, where X is a compact metric space, then the map $T : X \rightarrow X$ is almost one-to-one [19, Theorem 2.7].

Remark 2.1. Here we take the definition of almost one-to-one from [10, 19], as we shall use this one in Proposition 4.2 and in the construction of Example 3.6. Note that the denseness of $\pi^{-1}(Y_0)$ in X , used heavily in [3], is a sufficient but not

necessary condition; and these two conditions are equivalent, when π is a factor map between minimal systems (see Proposition 2.3).

A pair of points $x, y \in X$ is called *proximal* if $\liminf_{n \rightarrow \infty} \varrho(T^n x, T^n y) = 0$. In this case each of points from the pair is said to be *proximal* to another. Let $\pi : (X, T) \rightarrow (Y, S)$ be a factor map between dynamical systems. We call π *proximal* if any pair of points $x_1, x_2 \in X$ is proximal whenever $\pi(x_1) = \pi(x_2)$. Note that any almost one-to-one extension between minimal systems is a proximal extension.

Recall that the *natural extension* (\hat{X}, \hat{T}) of (X, T) is defined as

$$\hat{X} = \{(x_1, x_2, \dots) : T(x_{i+1}) = x_i \text{ and } x_i \in X \text{ for each } i \in \mathbb{N}\},$$

$$\hat{T} : (x_1, x_2, \dots) \mapsto (Tx_1, x_1, \dots),$$

with a compatible metric d given by

$$d((x_1, x_2, \dots), (x_1^*, x_2^*, \dots)) = \sum_{n \in \mathbb{N}} \frac{\varrho(x_n, x_n^*)}{2^n M} \text{ with } M = \text{diam}(X) + 1.$$

Then (\hat{X}, \hat{T}) is an invertible extension of (X, T) with a factor map $\hat{\pi} : (\hat{X}, \hat{T}) \mapsto (X, T)$, given by $(x_1, x_2, \dots) \mapsto x_1$. Observe that

$$\left(\prod_{i=1}^n U_i \times \prod_{n+1}^{\infty} X \right) \cap \hat{X} = \left(\prod_1^{n-1} X \times \bigcap_{i=1}^n T^{-(n-i)} U_i \times \prod_{n+1}^{\infty} X \right) \cap \hat{X}$$

for any opne $U_1, \dots, U_n \subset X$, and all such subsets form a basis for the topology of \hat{X} . Applying this fact, it is not hard to check from the definitions that (X, T) is minimal (sensitive, thickly sensitive, multi-sensitive, respectively) if and only if (\hat{X}, \hat{T}) is minimal (sensitive, thickly sensitive, multi-sensitive, respectively).

2.3. Other concepts. A continuous map $\phi : X \rightarrow Y$ is called *almost open* if $\phi(U)$ has a nonempty interior in Y for any opne $U \subset X$. Recall that if a system (X, T) is minimal then the map $T : X \rightarrow X$ is almost open [19]. It is easy to see that all of sensitivity, thick sensitivity and multi-sensitivity can be lifted from a factor to an extension by an almost open factor map by the method used in the proof of [14, Lemma 1.6]. Note that any factor map from a system containing a dense set of minimal points to a minimal system is almost open, as each factor map between minimal systems is also almost open [5, Theorem 1.15].

Lemma 2.2. *Let $\pi : (X, T) \rightarrow (Y, S)$ be a factor map between minimal systems. If $Y_1 \subset Y$ is a dense subset, then $\pi^{-1}(Y_1) \subset X$ is also a dense subset.*

Proof. If the conclusion does not hold, then $U \subset X$ is an opne subset, where U is the complement of the closure of $\pi^{-1}(Y_1)$ in X . Thus $\pi(U)$ has a nonempty interior in Y by [5, Theorem 1.15], and hence $\pi(U) \cap Y_1 \neq \emptyset$ by the denseness of Y_1 in Y , which implies $U \cap \pi^{-1}(Y_1) \neq \emptyset$, a contradiction with the construction of U . \square

Now we can prove the following

Proposition 2.3. *Let $\pi : (X, T) \rightarrow (Y, S)$ be a factor map between minimal systems. Denote by Y_0 the set of all points $y \in Y$ whose fibers $\pi^{-1}(y)$ are singletons. Then Y_0 is a dense subset of Y if and only if $\pi^{-1}(Y_0)$ is a dense subset of X .*

Proof. It suffices to show the denseness of $\pi^{-1}(Y_0)$ in X when Y_0 is dense in Y . Denote by Y_1 the set of all points $y \in Y$ such that $S^{-1}(y)$ is a singleton. Then $Y_1 \subset Y$ is a dense G_δ subset by [19, Theorem 2.7]; and hence so is $S^{-n}(Y_1) \subset Y$ for each $n \in \mathbb{Z}_+$, whose denseness in Y follows from Lemma 2.2. Now set

$$Y_* = Y_0 \cap Y_\infty \text{ with } Y_\infty = \bigcap_{n \geq 0} S^{-n}(Y_1).$$

We have that $Y_* \subset Y$ is a dense G_δ subset by the assumption of Y_0 .

Let $y \in Y_*$, and assume $\pi^{-1}(y) = \{x^*\}$ by the definition of Y_0 . It is clear that $Sy \in Y_\infty \subset Y_1$, in particular, $S^{-1}(Sy) = \{y\}$. Now if $x \in X$ satisfies $\pi(x) = Sy$, and take $x_0 \in T^{-1}(x)$, then $S(\pi x_0) = \pi(Tx_0) = Sy$, which implies $\pi(x_0) = y$ and hence $x_0 = x^*$, thus $x = Tx_0 = Tx^*$. That is, $\pi^{-1}(Sy) = \{Tx^*\}$, which implies $Sy \in Y_0$ and hence $Sy \in Y_*$. This shows that Y_* is a positively invariant subset of Y , and hence $\pi^{-1}(Y_*)$ is a positively invariant subset of X . Finally applying the minimality of the system (X, T) we obtain the denseness of $\pi^{-1}(Y_*)$ in X , and then the denseness of $\pi^{-1}(Y_0)$ in X . This finishes the proof. \square

We also use the following concepts by Furstenberg [12]. Let $\mathcal{S} \subset \mathbb{N}$. The set \mathcal{S} is called a *central set* if there exists a topological dynamical system (X, T) with $x \in X$ and open $U \subset X$ containing a minimal point y of (X, T) such that the pair (x, y) is proximal and $N_T(x, U) \subset \mathcal{S}$. The set \mathcal{S} is called a *difference set* or shortly Δ -set if there exists $\{s_1 < s_2 < \dots\} \subset \mathbb{N}$ with $\mathcal{S} = \{s_i - s_j : i > j\}$. The set \mathcal{S} is a Δ^* -set if it has a nonempty intersection with any Δ -set. Note that each central set contains a Δ -set [12, Proposition 8.10 and Lemma 9.1]; and if (X, T) is a minimal system, then $N_T(U, U)$ is a Δ^* -set for any opene $U \subset X$ by [12, Page 177].

3. DICHOTOMY OF MULTI-SENSITIVITY FOR TRANSITIVE SYSTEMS

The Auslander-Yorke dichotomy theorem states that a minimal system is either sensitive or equicontinuous (see [1, 2, 6, 14]). The goal of this section is to provide an analog of the Auslander-Yorke theorem for multi-sensitivity (see Theorem 3.1, Proposition 3.2 and Theorem 3.4), which is the main result of this paper.

Our dichotomy is stated firstly for minimal thickly sensitive systems as follows. Remark that recently Ye and Yu introduced and discussed block sensitivity and strong sensitivity for several families, and obtained results similar to Theorem 3.1 for these sensitivities [28]. We defer the long proof of it to Section 4.

Theorem 3.1. *Let (X, T) be a minimal system. Then (X, T) is not thickly sensitive if and only if (X, T) is an almost one-to-one extension of (X_{eq}, T_{eq}) .*

As shown by the following result, for transitive systems thick sensitivity is equivalent to multi-sensitivity. Observe that Moothathu pointed out firstly in [22] that multi-sensitivity implies thick sensitivity.

Proposition 3.2. *If (X, T) is multi-sensitive, then (X, T) is thickly sensitive. Moreover, if (X, T) is transitive, then the converse also holds.*

Proof. First assume that (X, T) is multi-sensitive with a sensitivity constant $\delta > 0$, and take any opene $U \subset X$. Let $k \in \mathbb{N}$. For each $i = 0, 1, \dots, k$, we choose opene $U_i \subset T^{-i}U$ such that $\max_{0 \leq j \leq k} \text{diam}(T^j U_i) < \delta$. By the assumption of δ we may select $n_k \in \bigcap_{i=0}^k S_T(U_i, \delta)$. Moreover, from the construction of U_0, U_1, \dots, U_k one has that

$n_k \in \bigcap_{i=0}^k S_T(T^{-i}U, \delta)$ and $n_k > k$. Obviously, $\{n_k - k, \dots, n_k - 1, n_k\} \subset S_T(U, \delta)$, which implies that (X, T) is thickly sensitive with a sensitivity constant δ .

Now we assume that a transitive system (X, T) is thickly sensitive with a sensitivity constant $\delta > 0$. Let $k \in \mathbb{N}$ and U_1, \dots, U_k be opene sets in X . Take a transitive point $x \in \text{Tran}(X, T)$. Then there exists $n_i \in \mathbb{N}$ such that $T^{n_i}x \in U_i$, where $i = 1, \dots, k$. So, we may get an opene $U \subset X$ such that $T^{n_i}U \subset U_i$ for every $i = 1, \dots, k$. By assumption there exists $s \in \mathbb{N}$ with $\{s, s+1, \dots, s+n_1+\dots+n_k\} \subset S_T(U, \delta)$, and then one has $s \in \bigcap_{i=1}^k S_T(U_i, \delta)$. This shows that (X, T) is multi-sensitive with a sensitivity constant δ . \square

Let (X, T) be an invertible system. Recall that $x \in X$ is an *almost automorphic point* of (X, T) if $T^{n_k}x \rightarrow x'$ implies $T^{-n_k}x' \rightarrow x$ for any $\{n_k : k \in \mathbb{N}\} \subset \mathbb{Z}$, where \mathbb{Z} is the set of all integers. The system (X, T) is said to be *almost automorphic* if $X = \overline{\text{orb}_T(x)}$ for an almost automorphic point $x \in X$. The structure of minimal almost automorphic systems was characterized in [24]: a minimal invertible system is almost automorphic if and only if it is an almost one-to-one extension of its maximal equicontinuous factor $(X_{\text{eq}}, T_{\text{eq}})$.

Thus, directly from Theorem 3.1, we have the following

Corollary 3.3. *Let (X, T) be an invertible minimal system. Then (X, T) is not multi-sensitive if and only if it is almost automorphic.*

We are going to link thick sensitivity with local equicontinuity of points by introducing the concept of syndetically equicontinuous points of a system.

We say that $x \in X$ is *syndetically equicontinuous* if for any $\varepsilon > 0$ there exists a neighborhood U of x such that $J_T(U, \varepsilon)$ is a syndetic set. Denote by $\text{Eq}_{\text{syn}}(X, T)$ the set of all syndetically equicontinuous points of (X, T) . Then $\text{Eq}_{\text{syn}}(X, T) \supset \text{Eq}(X, T)$. Since a thick set has a nonempty intersection with every syndetic set, one has readily that if (X, T) is thickly sensitive then $\text{Eq}_{\text{syn}}(X, T) = \emptyset$, equivalently, if $\text{Eq}_{\text{syn}}(X, T) \neq \emptyset$ then (X, T) is not thickly sensitive.

Recall the *Auslander-Yorke Dichotomy Theorem* from [6] as follows, supplemented by some results from [14] and [2].

Auslander-Yorke Dichotomy Theorem. *Let (X, T) be a transitive system. Then exactly one of the following two cases holds.*

$\text{Eq}(X, T) \neq \emptyset$: Assume that there exists an equicontinuity point for the system. *The equicontinuity points are exactly the transitive points, i.e., $\text{Eq}(X, T) = \text{Tran}(X, T)$, and the system is almost equicontinuous. The map T is a homeomorphism and the inverse system (X, T^{-1}) is also almost equicontinuous. Furthermore, the system is uniformly rigid meaning that some subsequence of $\{T^n : n = 0, 1, \dots\}$ converges uniformly to the identity map on X .*

$\text{Eq}(X, T) = \emptyset$: Assume that the system has no equicontinuity points. *The system is sensitive.*

Similarly, we have the following dichotomy. We call the system (X, T) *syndetically equicontinuous* if $\text{Eq}_{\text{syn}}(X, T) = X$.

Theorem 3.4. *Let (X, T) be a transitive system. Then either (X, T) is thickly sensitive and so $\text{Eq}_{\text{syn}}(X, T) = \emptyset$, or $\text{Tran}(X, T) \subset \text{Eq}_{\text{syn}}(X, T)$. In particular, if (X, T) is minimal then it is either thickly sensitive or syndetically equicontinuous.*

Proof. It suffices to show that, if (X, T) is not thickly sensitive then $\text{Tran}(X, T) \subset \text{Eq}_{\text{syn}}(X, T)$. Let $\delta > 0$. By the assumption there exists opene $U' \subset X$ such that $S_T(U', \delta)$ is not thick. Equivalently, there exists syndetic $\mathcal{N} \subset \mathbb{N}$ such that $\varrho(T^n x_1, T^n x_2) \leq \delta$ whenever $x_1, x_2 \in U'$ and $n \in \mathcal{N}$. Now for any $x \in \text{Tran}(X, T)$ there exists $m \in \mathbb{N}$ with $T^m x \in U'$ and hence there exists open $U \subset X$ containing x with $T^m U \subset U'$. In particular, $\varrho(T^{m+n} x, T^{m+n} x') \leq \delta$ whenever $x' \in U$ and $n \in \mathcal{N}$. That implies $x \in \text{Eq}_{\text{syn}}(X, T)$, as $m + \mathcal{N}$ is a syndetic set. \square

Note that the dichotomy Theorem 3.4 does not work when the system (X, T) is not transitive. The following example is due to an anonymous referee of the paper.

Example 3.5. *There exists a system (X, T) , which is not thickly sensitive, such that $\text{Eq}_{\text{syn}}(X, T) = \emptyset$.*

Construction. Let (Y_1, S_1) be the full shift homeomorphism on two symbols, that is, $Y_1 = \{0, 1\}^{\mathbb{Z}}$ and $S_1 : (y_i : i \in \mathbb{Z}) \mapsto (y_{i+1} : i \in \mathbb{Z})$. We take a fixed point $e_1 \in Y_1$. Let (Y_2, S_2) be the identity map on the one-point compactification of the discrete space \mathbb{N} with $e_2 \in Y_2$ the point at infinity. Now we set (X, T) to be the factor of the product system $(Y_1 \times Y_2, S_1 \times S_2)$ by collapsing $(Y_1 \times \{e_2\}) \cup (\{e_1\} \times Y_2)$ into a fixed point $e \in X$. Then (X, T) is the required system:

By the construction of X , for each $\delta > 0$, there exists $p \in \mathbb{N}$ such that $Y_1 \times \{p\} \subset X$ is an open invariant subset with $\text{diam}(Y_1 \times \{p\}) < \delta$, in particular, the set $S_T(Y_1 \times \{p\}, \delta) = \emptyset$ is not thick. This implies that (X, T) is not thickly sensitive.

Let $x \in X$. By the construction of X , there exists $q \in \mathbb{N}$ such that any neighborhood of x has a nonempty intersection with $Y_1 \times \{q\}$, and set $\delta_x = \frac{1}{2} \text{diam}(Y_1 \times \{q\}) > 0$. It is easy to check that, for each neighborhood U of x there exists $N_U \in \mathbb{N}$ such that $\text{diam}(T^n U) > \delta_x$ for all $n > N_U$, in particular, the set $J_T(U, \delta_x) \subset \{1, \dots, N_U\}$ is not syndetic. This shows $x \notin \text{Eq}_{\text{syn}}(X, T)$, and then $\text{Eq}_{\text{syn}}(X, T) = \emptyset$. \square

The following examples show that for a transitive not thickly sensitive system (X, T) the nonempty set $\text{Eq}_{\text{syn}}(X, T)$ may be very complicated. Note that by [14, Theorem 1.3] each non sensitive E-system is necessarily a minimal equicontinuous system.

Example 3.6. *There exists a sensitive, not thickly sensitive system (X, T) such that $\text{Tran}(X, T) \subsetneq \text{Eq}_{\text{syn}}(X, T) = X$. In fact, the constructed system (X, T) is a nonminimal E-system.*

Construction. We take a Toeplitz flow (Y, S) which is a minimal invertible system with positive topological entropy. See [10] for the construction of such a system. Let $\pi_{\text{eq}} : (Y, S) \rightarrow (Y_{\text{eq}}, S_{\text{eq}})$ be the factor map of (Y, S) over its maximal equicontinuous factor. Then π_{eq} is an almost one-to-one extension between minimal systems. And hence (Y, S) is not thickly sensitive by Theorem 3.1, and π_{eq} is a proximal extension.

By the classical variational principle (see for example [27, Theorem 8.6]) we choose an ergodic invariant Borel probability measure ν of (Y, S) with positive measure-theoretic ν -entropy $h_\nu(Y, S)$. Let $\nu = \int_Z \nu_z d\eta(z)$ be the disintegration of ν over the Pinsker factor $(Z, \mathcal{D}, \eta, R)$ of $(Y, \mathcal{B}_Y, \nu, S)$, where (Y, \mathcal{B}_Y, ν) is the completion of (Y, \mathcal{B}_Y, ν) and \mathcal{B}_Y denotes the Borel σ -algebra of Y (for the construction of such a disintegration see for example [12, Chapter 5, §4]). Set $\lambda = \int_Z \nu_z \times \nu_z d\eta(z)$, which is in fact an ergodic invariant Borel probability measure of $(Y \times Y, S \times S)$ with positive measure-theoretic λ -entropy and $\lambda(X \setminus \Delta_Y) > 0$ by [13], where $\Delta_Y = \{(y, y) : y \in Y\}$ and $X \subset Y \times Y$ is the support of λ , that is, X is the smallest closed

subset of $Y \times Y$ with $\lambda(X) = 1$. It is easy to see that $(X, S \times S)$ forms a transitive system having a nonempty intersection with Δ_Y , denoted by (X, T) . Then $X \supsetneq \Delta_Y$ by the minimality of (Y, S) . Additionally, if $(y_1, y_2) \in X$ then $\pi_{\text{eq}}(y_1) = \pi_{\text{eq}}(y_2)$, and hence y_1 and y_2 are proximal (as π_{eq} is a proximal extension). In particular, if $(y_1, y_2) \in X$ is a minimal point of (X, T) then $y_1 = y_2$, and so Δ_Y is the unique minimal subsystem contained in (X, T) .

Now we check that the system (X, T) satisfies all required properties. Clearly, (X, T) is a nonminimal E-system. Moreover, it is an almost one-to-one extension of a minimal equicontinuous system $(Y_{\text{eq}}, S_{\text{eq}})$ (because π_{eq} is almost one-to-one and $\pi_{\text{eq}}(y_1) = \pi_{\text{eq}}(y_2)$ for all $(y_1, y_2) \in X$), and hence not thickly sensitive by Proposition 4.2. In fact, we can also obtain it directly from the definitions.

Now let us check $\text{Eq}_{\text{syn}}(X, T) = X$. Let $x \in X$ and $y_0 \in Y$, and hence $y_0 \in \text{Eq}_{\text{syn}}(Y, S)$ by Theorem 3.4. Take a compatible metric ϱ_1 over Y , and a compatible metric $\varrho((y_1, y_2), (y'_1, y'_2)) = \max\{\varrho_1(y_1, y'_1), \varrho_1(y_2, y'_2)\}$ over X for $y_1, y'_1, y_2, y'_2 \in Y$. Then for each $\delta > 0$ there exists open $U_Y \subset Y$ containing y_0 and syndetic $\mathcal{N} \subset \mathbb{N}$ such that $\varrho_1(S^n y, S^n y_0) \leq \delta$ whenever $y \in U_Y$ and $n \in \mathcal{N}$. Thus $\varrho_1(S^n y_1, S^n y_2) \leq 2\delta$ whenever $y_1, y_2 \in U_Y$ and $n \in \mathcal{N}$, and finally $\varrho(T^n x_1, T^n x_2) \leq 2\delta$ whenever $x_1, x_2 \in (U_Y \times U_Y) \cap X$ and $n \in \mathcal{N}$. As Δ_Y is the unique minimal subsystem of (X, T) , there exist $m \in \mathbb{N}$ and open $U \subset X$ containing x with $T^m U \subset (U_Y \times U_Y) \cap X$, and so $\varrho(T^{m+n} x, T^{m+n} x') \leq 2\delta$ whenever $x' \in U$ and $n \in \mathcal{N}$. This implies $x \in \text{Eq}_{\text{syn}}(X, T)$, because $m + \mathcal{N} \subset \mathbb{N}$ is syndetic. The construction is done. \square

Example 3.7. *There exists a nonminimal E-system (X', T') which is not thickly sensitive, such that $\text{Tran}(X', T') \subsetneq \text{Eq}_{\text{syn}}(X', T') \subsetneq X'$.*

Construction. Let (X, T) and (Y, S) be the systems as constructed in Example 3.6, and we take (X', T') to be the system constructed by collapsing Δ_Y into a fixed point p_0 of (X, T) . Then the fixed point p_0 is the unique minimal point of (X', T') . Let $\pi : (X, T) \rightarrow (X', T')$ be the corresponding factor map.

Now we check that the system (X', T') satisfies all required properties. It is easy to see that (X', T') is an invertible nonminimal E-system. Then $X' \setminus \text{Tran}(X', T')$ is a dense subset of X' (see [18]), and hence $X' \setminus \text{Tran}(X', T') \supsetneq \{p_0\}$. Moreover, the factor map π is almost open. This implies that (X', T') is not thickly sensitive, because (X, T) is not thickly sensitive. In fact, $\pi : X \setminus \Delta_Y \rightarrow X' \setminus \{p_0\}$ is a homeomorphism. Therefore we obtain that $\text{Tran}(X', T') \subsetneq X' \setminus \{p_0\} \subset \text{Eq}_{\text{syn}}(X', T')$, as $\text{Eq}_{\text{syn}}(X, T) = X$.

Finally we are going to show that $p_0 \notin \text{Eq}_{\text{syn}}(X', T')$ and hence $\text{Tran}(X', T') \subsetneq X' \setminus \{p_0\} = \text{Eq}_{\text{syn}}(X', T') \subsetneq X'$. Let $x_* \in X' \setminus \{p_0\}$ and set $0 < \delta < \text{dist}(\{x_*\}, \{p_0\})$. Let $U_0 \subset X$ be any open set containing p_0 and $m \in \mathbb{N}$. By shrinking U_0 we may choose open U_* containing x_* such that $\text{dist}(U_*, U_0) > \delta$. We take open $W \subset U_0$ such that $(T')^{-j} W \subset U_0$ for all $j = 0, 1, \dots, m$ (as $T'p_0 = p_0$), and then

$$\begin{aligned} N_{(T')^{-1}}(U_*, U_0) &\supset \bigcup_{j=0}^m N_{(T')^{-1}}(U_*, (T')^{-j} W) \\ &\supset \{n + j : n \in N_{(T')^{-1}}(U_*, W), j = 0, 1, \dots, m\}. \end{aligned}$$

Thus $N_{T'}(U_0, U_*) = N_{(T')^{-1}}(U_*, U_0)$ is a thick set, because $N_{(T')^{-1}}(U_*, W) \neq \emptyset$ by the transitivity of $(X', (T')^{-1})$ (as (X', T') is an invertible transitive system, $(X', (T')^{-1})$ is also transitive from the definition). And hence $S_{T'}(U_0, \delta)$ is a thick set, as $T'p_0 = p_0$. In particular, $p_0 \notin \text{Eq}_{\text{syn}}(X', T')$. The construction is done. \square

Theorem 3.4 from [19] says that any irrational rotation of the two dimensional torus has an almost one-to-one extension which is a noninvertible minimal map of that torus. By Proposition 4.2 any of these above mentioned noninvertible minimal maps on the torus is syndetically equicontinuous, and by the Auslander-Yorke Dichotomy Theorem any noninvertible minimal map is sensitive. But we still have the following open question.

Question. *If a syndetically equicontinuous system is uniformly rigid then is it equicontinuous?*

4. PROOF OF THEOREM 3.1

In this section we present a proof of our dichotomy Theorem 3.1.

Lemma 4.1. *Let $\pi : (X, T) \rightarrow (Y, S)$ be a factor map and let $y_0 \in \text{Eq}(Y, S)$ be a minimal point such that $\pi^{-1}(y_0) = \{x_0\}$. Then $x_0 \in \text{Eq}_{\text{syn}}(X, T)$. In particular, (X, T) is not thickly sensitive.*

Proof. Let $\delta > 0$ and let ϱ_Y be a compatible metric over Y . We take open $W \subset X$ containing x_0 such that $\text{diam}(W) < \delta$. As $\{x_0\} = \pi^{-1}(y_0)$, there exists open $V \subset Y$ containing y_0 such that $\pi^{-1}(V) \subset W$. Let $\varepsilon > 0$ be small enough such that $\{y \in Y : \varrho_Y(y_0, y) < 2\varepsilon\} \subset V$. Since $y_0 \in \text{Eq}(Y, S)$, there exists $\varepsilon \geq \kappa > 0$ such that $\varrho_Y(S^n y, S^n y_0) < \varepsilon$ whenever $\varrho_Y(y, y_0) < \kappa$ and $n \in \mathbb{N}$. Take $V' = \{y \in Y : \varrho_Y(y_0, y) < \kappa\}$, $U = \pi^{-1}(V') \ni x_0$ and set $\mathcal{S} = N_S(y_0, V')$. Note that \mathcal{S} is syndetic, because y_0 is a minimal point. Let $n \in \mathcal{S}$. Then $S^n y_0 \in V'$, additionally, if $y \in V'$ then $\varrho_Y(S^n y_0, S^n y) < \varepsilon$ and so $\varrho_Y(y_0, S^n y) < 2\varepsilon$, that gives $S^n V' \subset V$, and hence

$$T^n U = T^n \pi^{-1}(V') \subset \pi^{-1}(S^n V') \subset \pi^{-1}(V) \subset W.$$

Summing up, for each $\delta > 0$ there exist open $U \subset X$ containing x_0 and a syndetic set $\mathcal{S} \subset \mathbb{N}$ such that $\varrho(T^n x_0, T^n x) < \delta$ for all $x \in U$ and any $n \in \mathcal{S}$ (recall that ϱ is the metric over X). That is, $x_0 \in \text{Eq}_{\text{syn}}(X, T)$. \square

Similarly, we can provide proof of the following result, which is in fact Lemma 6.3 from [16].

Proposition 4.2. *If (X, T) is an almost one-to-one extension of a minimal equicontinuous system, then it is syndetically equicontinuous.*

In order to prove the another direction of Theorem 3.1, we make some preparations. Let $\pi : (X, T) \rightarrow (Y, S)$ be a factor map between dynamical systems. Recall that π is proximal if any pair of points $x_1, x_2 \in X$ is proximal for (X, T) whenever $\pi(x_1) = \pi(x_2)$. The following result should be well known, but we fail to find a reference and hence provide a proof of it here for completeness. This variant of a very short proof was kindly communicated to us by Joseph Auslander and works for systems of any acting group.

Lemma 4.3. *Let $\pi_i : (X_i, T_i) \rightarrow (Y_i, S_i)$, $i = 1, 2$ be factor maps between dynamical systems. If both π_1 and π_2 are proximal, then the product factor map $\pi_1 \times \pi_2 : (X_1 \times X_2, T_1 \times T_2) \rightarrow (Y_1 \times Y_2, S_1 \times S_2)$ is also proximal.*

Proof. Let a pair of points $(x_1, x_2), (x_1^*, x_2^*) \in X_1 \times X_2$ with $\pi_i(x_i) = \pi_i(x_i^*)$ for $i = 1, 2$. We will show that this pair is proximal. In fact, take a minimal point $((z_1, z_2), (z_1^*, z_2^*))$ from the orbit closure of $((x_1, x_2), (x_1^*, x_2^*))$ in the system $(X_1 \times X_2 \times X_1 \times X_2, T_1 \times T_2 \times T_1 \times T_2)$. Then (z_1, z_1^*) is a minimal point in the system

$(X_1 \times X_1, T_1 \times T_1)$, and $\pi_1(z_1) = \pi_1(z_1^*)$ and hence (z_1, z_1^*) is proximal, which implies $z_1 = z_1^*$. Similarly $z_2 = z_2^*$. In particular, (x_1, x_2) and (x_1^*, x_2^*) are proximal. \square

Then we have the following

Proposition 4.4. *Let $\pi : (X, T) \rightarrow (Y, S)$ be a factor, not almost one-to-one map between minimal systems, where (Y, S) is invertible. Then $\inf_{y \in Y} \text{diam}(\pi^{-1}y) > 0$. Moreover, if π is also proximal, then (X, T) is thickly sensitive.*

Proof. As (Y, S) is an invertible minimal system, it is not hard to show that $\pi^{-1}(y)$ is not a singleton for any $y \in Y$. Let us first prove that $d := \inf_{y \in Y} \text{diam}(\pi^{-1}y) > 0$.

Let $\psi : Y \rightarrow [0, \text{diam}(X)]$ be given by $y \mapsto \text{diam}(\pi^{-1}y)$, and hence for each $y \in Y$ one has $\psi(y) > 0$ as $\pi^{-1}(y)$ is not a singleton. Since the function ψ is upper semi-continuous, $E_c(\psi)$ - the set of all points of continuity of ψ , is a residual subset of Y (see for example [12, Lemma 1.28]). Suppose that $d = 0$. So, there exists a sequence of points $y_i \in Y$ such that $\lim_{i \rightarrow \infty} \psi(y_i) = 0$.

Let $y_c \in E_c(\psi)$ and $\varepsilon > 0$. There exists open $V \subset Y$ containing y_c such that $|\psi(y_c) - \psi(y)| \leq \varepsilon$ whenever $y \in V$. Since (Y, S) is minimal, there exists $m \in \mathbb{N}$ with $\bigcup_{j=0}^m S^{-j}V = Y$. By taking a subsequence (if necessary) we may assume that $\{y_i : i \in \mathbb{N}\} \subset S^{-k}V$ for some $k \in \{0, 1, \dots, m\}$. Since $\lim_{i \rightarrow \infty} \psi(y_i) = 0$, in other words, the diameter of $\pi^{-1}(y_i)$ tends to zero, the diameter of $\pi^{-1}(S^k y_i) = T^k \pi^{-1}(y_i)$ also tends to zero. Therefore $\lim_{i \rightarrow \infty} \psi(S^k y_i) = 0$, which implies that $\psi(y_c) \leq \varepsilon$ by the construction of V and m . Finally $\psi(y_c) = 0$, a contradiction.

Take $0 < \delta < \frac{d}{6}$. Now assume that π is proximal. We shall prove that (X, T) is thickly sensitive with a sensitivity constant δ . Let $x_* \in X$ and $m \in \mathbb{N}$ and take open $U \subset X$ containing x_* . Let $V \subset U$ be an open set containing x_* with $\max_{0 \leq i \leq m} \text{diam}(T^i V) < \delta$. Since a factor map between minimal systems is almost open [5, Theorem 1.15], therefore for each $i = 0, 1, \dots, m$ we can choose $y_i \in \text{int}(\pi(T^i V))$ (the interior of $\pi(T^i V)$), $u_i \in \pi^{-1}(y_i)$ with $\text{dist}(u_i, T^i V) > \frac{d}{2} - \delta$ (because $\text{diam}(T^i V) < \delta$), and set

$$W_i = \{x \in X : \varrho(x, u_i) < \delta\} \cap \pi^{-1}(\text{int}(\pi(T^i V))) \ni u_i.$$

Obviously, $\text{dist}(W_i, T^i V) > \frac{d}{2} - 2\delta > \delta$ for each $i = 0, 1, \dots, m$.

Note that since (X, T) is minimal, the set of all minimal points of the system $(X^{m+1}, T^{(m+1)})$, the product system of $m+1$ copies of (X, T) , is dense in X^{m+1} . Hence we can take a minimal point $(v_0, v_1, \dots, v_m) \in W_0 \times W_1 \times \dots \times W_m$ of the system $(X^{m+1}, T^{(m+1)})$, and let $x_i \in T^i V$ with $\pi(x_i) = \pi(v_i)$ (because $\pi(v_i) \in \pi(T^i V)$) for each $i = 0, 1, \dots, m$. Since the factor map $\pi : (X, T) \rightarrow (Y, S)$ is proximal, by Lemma 4.3,

$$\pi' : (X^{m+1}, T^{(m+1)}) \rightarrow (Y^{m+1}, S^{(m+1)}), (x'_i : 0 \leq i \leq m) \mapsto (\pi(x'_i) : 0 \leq i \leq m)$$

is also a proximal factor map. In particular, $((x_0, x_1, \dots, x_m), (v_0, v_1, \dots, v_m))$ is proximal (under the action $T^{(m+1)}$), and thus

$$S = N_{T^{(m+1)}}((x_0, x_1, \dots, x_m), W_0 \times W_1 \times \dots \times W_m)$$

is a central set and contains a Δ -set by [12]. Finally $\mathcal{S} \cap \mathcal{N} \neq \emptyset$ where $\mathcal{N} = N_T(V, V) \subset N_T(TV, TV) \subset \dots \subset N_T(T^m V, T^m V)$ is a Δ^* -set [12]. Now for any $n \in \mathcal{S} \cap \mathcal{N}$ and each $i = 0, 1, \dots, m$, on one hand $T^n x_i \in W_i$ as $n \in \mathcal{S}$, and hence $T^{n+i} V \cap W_i \ni T^n x_i$ as $x_i \in T^i V$; on the other hand $T^{n+i} V \cap T^i V \neq \emptyset$ as $n \in \mathcal{N}$, therefore $\text{diam}(T^{n+i} V) \geq \text{dist}(W_i, T^i V) > \delta$. Thus

$$S_T(U, \delta) \supset S_T(V, \delta) \supset \{n + i : n \in \mathcal{S} \cap \mathcal{N}, i = 0, 1, \dots, m\},$$

which implies that (X, T) is thickly sensitive by the arbitrariness of U and m . \square

The following lemma is just a reformulation of [25, Theorem 1.1].

Lemma 4.5. *Let (X, T) be an invertible minimal system and $x, x' \in X$. Then $(x, x') \in Q_+(X, T)$ if and only if for every open $U, V \subset X$ containing x and x' , respectively, there exist $n_1, m_1 \in \mathbb{Z}$ such that $T^{n_1} x, T^{n_1+m_1} x \in U$ and $T^{m_1} x \in V$.*

Now let us show that it is also true for any (not only invertible) continuous minimal map. Recall that $\mathcal{S} \subset \mathbb{N}$ is an *IP set* if there exists $\{p_k : k \in \mathbb{N}\} \subset \mathbb{N}$ with $\{p_{i_1} + \dots + p_{i_k} : k \in \mathbb{N} \text{ and } i_1 < \dots < i_k\} \subset \mathcal{S}$.

Lemma 4.6. *Let (X, T) be a minimal system and $x, y \in X$. Then $(x, y) \in Q_+(X, T)$ if and only if for every open $U, V \subset X$ containing x and y , respectively, there exist $n, m \in \mathbb{N}$ such that $T^n x, T^{n+m} x \in U$ and $T^m x \in V$.*

Proof. Firstly assume that, for every open $U, V \subset X$ containing x and y , respectively, there exist $n, m \in \mathbb{N}$ such that $T^n x, T^{n+m} x \in U$ and $T^m x \in V$. From the definition it is readily to obtain $(x, y) \in Q_+(X, T)$, for instance by taking $x' = x, y' = T^m(x)$.

Now assume $(x, y) \in Q_+(X, T)$ and take open $U, V \subset X$ containing x and y , respectively. Recall that (\hat{X}, \hat{T}) is the natural extension of (X, T) and $\hat{\pi} : (\hat{X}, \hat{T}) \rightarrow (X, T)$ is the associated factor map. Hence $Q_+(X, T) = (\hat{\pi} \times \hat{\pi})Q_+(\hat{X}, \hat{T})$ by [17, Lemma A.3]. In particular, there exist $(x_*, y_*) \in Q_+(\hat{X}, \hat{T})$ and open $U_*, V_* \subset \hat{X}$ containing x_* and y_* , respectively, such that $\hat{\pi}(x_*) = x, \hat{\pi}(y_*) = y$ and $\hat{\pi}(U_*) \subset U, \hat{\pi}(V_*) \subset V$. Since (X, T) is minimal, (\hat{X}, \hat{T}) is an invertible minimal system, and then by applying Lemma 4.5 there exist $n_1, m_1 \in \mathbb{Z}$ such that $\hat{T}^{n_1} x_*, \hat{T}^{n_1+m_1} x_* \in U_*$ and $\hat{T}^{m_1} x_* \in V_*$. Moreover, we choose open $W \subset \hat{X}$ containing x_* such that $\hat{T}^{n_1} W \subset U_*, \hat{T}^{n_1+m_1} W \subset U_*$ and $\hat{T}^{m_1} W \subset V_*$. Since (\hat{X}, \hat{T}) is minimal, x_* is recurrent in the sense that $\hat{T}^{l_k} x_*$ tends to x_* for a sequence of positive integers $l_1 < l_2 < \dots$, and so $N_{\hat{T}}(x_*, W)$ is an IP set by [12, Theorem 2.17]. Hence there exist $p_1, q_1 \in \mathbb{N}$ such that

$$n = n_1 + p_1 > 0, m = m_1 + q_1 > 0 \text{ and } \{p_1, q_1, p_1 + q_1\} \subset N_{\hat{T}}(x_*, W).$$

Thus $\hat{T}^n x_*, \hat{T}^{n+m} x_* \in U_*$ and $\hat{T}^m x_* \in V_*$. Therefore $T^n x, T^{n+m} x \in U$ and $T^m x \in V$ by the above construction. This finishes the proof. \square

With the help of Lemma 4.6, using an idea of the proof of [24, Lemma 2.1.2] we obtain the following result, which is of independent interest.

Proposition 4.7. *Let (X, T) be a minimal system and $x, y \in X$. Then $(x, y) \in Q_+(X, T)$ if and only if $N_T(x, U)$ contains a Δ -set for any open $U \subset X$ containing y .*

Proof. Sufficiency. Let $U \subset X$ be an open set containing y . Since $N_T(x, U)$ contains a Δ -set by the assumption, there exists $\{s_1 < s_2 < s_3\} \subset \mathbb{N}$ with $T^{s_3-s_2}x, T^{s_2-s_1}x, T^{s_3-s_1}x \in U$. Let $x' = x, y' = T^{s_2-s_1}x$ and $m = s_3 - s_2 \in \mathbb{N}$. Then $T^m x', T^m y' \in U$ and $(x, y) \in Q_+(X, T)$ by the arbitrariness of U .

Necessity. Assume $(x, y) \in Q_+(X, T)$ and take open $U \subset X$ containing y . Choose positive real numbers η and $\eta_k, k \in \mathbb{N}$ such that $\eta = \sum_{k \in \mathbb{N}} \eta_k$ and $B_\eta(y) \subset U$,

where $B_\eta(y)$ denotes the open ball of radius η centered at y . By applying Lemma 4.6 to $B_{\eta_1}(x)$ and $B_{\eta_1}(y)$, there exist $n_1, m_1 \in \mathbb{N}$ such that

$$T^{n_1}x, T^{n_1+m_1}x \in B_{\eta_1}(x) \text{ and } T^{m_1}x \in B_{\eta_1}(y).$$

Fix a $\delta > 0$. Applying Lemma 4.6 to $B_\delta(x)$ and $B_{\eta_1}(y)$, we have $n_2, m_2 \in \mathbb{N}$ such that $T^{n_2}x, T^{n_2+m_2}x \in B_\delta(x)$ and $T^{m_2}x \in B_{\eta_2}(y)$. Since δ can be selected small enough, we can require additionally

$$\max_{0 \leq r \leq n_1+m_1} \varrho(T^{r+n_2}x, T^r x) < \eta_2 \text{ and } \max_{0 \leq r \leq n_1+m_1} \varrho(T^{r+n_2+m_2}x, T^r x) < \eta_2.$$

We continue the process by induction. Put $l_k = \sum_{i=1}^k (n_i + m_i)$ for each $k \in \mathbb{N}$. Then there exist $n_{k+1}, m_{k+1} \in \mathbb{N}$ such that $T^{m_{k+1}}x \in B_{\eta_{k+1}}(y)$,

$$(4.1) \quad \max_{0 \leq r \leq l_k} \varrho(T^{r+n_{k+1}}x, T^r x) < \eta_{k+1} \text{ and } \max_{0 \leq r \leq l_k} \varrho(T^{r+n_{k+1}+m_{k+1}}x, T^r x) < \eta_{k+1}.$$

Set $p_k = m_k + n_{k+1}$ and $s_k = p_1 + \dots + p_k$ for every $k \in \mathbb{N}$. Then, for all $i \leq j$,

$$\begin{aligned} \varrho(T^{p_i+\dots+p_j}x, y) &= \varrho(T^{m_i+\sum_{k=i+1}^j (n_k+m_k)+n_{j+1}}x, y) \\ &\leq \varrho(T^{m_i+\sum_{k=i+1}^j (n_k+m_k)+n_{j+1}}x, T^{m_i+\sum_{k=i+1}^j (n_k+m_k)}x) + \\ &\quad \dots + \varrho(T^{m_i+(n_{i+1}+m_{i+1})}x, T^{m_i}x) + \varrho(T^{m_i}x, y) \\ &< \eta_{j+1} + \dots + \eta_i < \eta \text{ (using (4.1)).} \end{aligned}$$

So, $N_T(x, U)$ contains the Δ -set $\{s_j - s_i : i < j\}$ from the construction. \square

Recall that $\pi_{\text{eq}} : (X, T) \rightarrow (X_{\text{eq}}, T_{\text{eq}})$ is the corresponding factor map of (X, T) over its maximal equicontinuous factor.

Proposition 4.8. *Let (X, T) be a minimal system. Assume that $\pi_{\text{eq}} : (X, T) \rightarrow (X_{\text{eq}}, T_{\text{eq}})$ is not proximal. Then (X, T) is thickly sensitive.*

Proof. Since $\pi_{\text{eq}} : (X, T) \rightarrow (X_{\text{eq}}, T_{\text{eq}})$ is not proximal, there exists a pair of points $x_1, x_2 \in X$, which is not proximal, such that $\pi_{\text{eq}}(x_1) = \pi_{\text{eq}}(x_2)$ (and hence $(x_1, x_2) \in Q_+(X, T)$, as (X, T) is minimal). Then $d := \inf_{n \in \mathbb{N}} \varrho(T^n x_1, T^n x_2) > 0$.

Take $0 < \delta < \frac{d}{3}$. We are going to prove that (X, T) is thickly sensitive with a sensitivity constant $\delta > 0$. Since (X, T) is minimal, it suffices to show that $S_T(U, \delta)$ is thick for any open $U \subset X$ containing x_1 .

For any $m \in \mathbb{N}$ take open sets $V \subset U$ and W containing x_1 and x_2 , respectively, such that $\max_{0 \leq i \leq m} \max\{\text{diam}(T^i V), \text{diam}(T^i W)\} < \delta$. By the above construction $\min_{0 \leq i \leq m} \text{dist}(T^i V, T^i W) > \delta$. Since $(x_1, x_2) \in Q_+(X, T)$, $N_T(x_1, W)$ contains a Δ -set by Proposition 4.7, and hence has a nonempty intersection with \mathcal{N} , where

$\mathcal{N} = N_T(V, V) \subset N_T(TV, TV) \subset \cdots \subset N_T(T^m V, T^m V)$ is a Δ^* -set by [12, Page 177]. Therefore for every $n \in N_T(x_1, W) \cap \mathcal{N}$ and $i = 0, 1, \dots, m$ we have: $T^{n+i}V \cap T^i W \ni T^{n+i}x_1$, because $T^n x_1 \in W$; and $T^{n+i}V \cap T^i V \neq \emptyset$, because $n \in \mathcal{N}$. That gives $\text{diam}(T^{n+i}V) \geq \text{dist}(T^i W, T^i V) > \delta$. Thus

$$S_T(U, \delta) \supset S_T(V, \delta) \supset \{n + i : n \in N_T(x_1, W) \cap \mathcal{N}, i = 0, 1, \dots, m\},$$

which implies that (X, T) is thickly sensitive. \square

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. If π_{eq} is almost one-to-one, then (X, T) is not thickly sensitive by Proposition 4.2. Now assume that (X, T) is not thickly sensitive, then π_{eq} is proximal by Proposition 4.8, and then π_{eq} is almost one-to-one by Proposition 4.4 (as $(X_{\text{eq}}, T_{\text{eq}})$ is an invertible minimal system). This finishes the proof. \square

As a corollary of Theorem 3.1, we have the following

Proposition 4.9. *Let $\pi : (X, T) \rightarrow (Y, S)$ be an almost one-to-one factor map between minimal systems. Then (X, T) is syndetically equicontinuous if and only if so is (Y, S) .*

Proof. By Theorem 3.4, it suffices to prove that (X, T) is not thickly sensitive if and only if so is (Y, S) . As a factor map between minimal systems, π is almost open by [5, Theorem 1.15], and so if (X, T) is not thickly sensitive then so is (Y, S) , as the thick sensitivity can be lifted from a factor to an extension by an almost open factor map by the method used in the proof of [14, Lemma 1.6].

Now assume that (Y, S) is not thickly sensitive, and then the factor map $\pi_{\text{eq}}^* : (Y, S) \rightarrow (Y_{\text{eq}}, S_{\text{eq}})$, the factor map of (Y, S) over its maximal equicontinuous factor $(Y_{\text{eq}}, S_{\text{eq}})$, is almost one-to-one by Theorem 3.1. Set π^* to be the composition factor map $\pi_{\text{eq}}^* \circ \pi : (X, T) \rightarrow (Y_{\text{eq}}, S_{\text{eq}})$. Denote by Y_1 (Y_2 , Y_0 , respectively) the set of all points $y_1 \in Y_{\text{eq}}$ ($y_2 \in Y_{\text{eq}}$, $y_0 \in Y$, respectively) whose fibers $(\pi^*)^{-1}(y_1)$ ($(\pi_{\text{eq}}^*)^{-1}(y_2)$, $\pi^{-1}(y_0)$, respectively) are singletons. Then Y_2 is a dense G_δ subset of Y_{eq} , and Y_0 is a dense G_δ subset of Y . This implies that $Y_0 \cap (\pi_{\text{eq}}^*)^{-1}(Y_2)$ is a dense G_δ subset of Y by Lemma 2.2, and then $\pi_{\text{eq}}^*(Y_0 \cap (\pi_{\text{eq}}^*)^{-1}(Y_2))$ is a dense subset of Y_{eq} . Note that $\pi_{\text{eq}}^*(Y_0 \cap (\pi_{\text{eq}}^*)^{-1}(Y_2)) \subset Y_1$. In fact, for any $y_* \in \pi_{\text{eq}}^*(Y_0 \cap (\pi_{\text{eq}}^*)^{-1}(Y_2)) \subset Y_2$, we take $y_0 \in Y_0$ with $\pi_{\text{eq}}^*(y_0) = y_*$, then $(\pi_{\text{eq}}^*)^{-1}(y_*) = \{y_0\}$ as $y_* \in Y_2$, and hence

$$(\pi^*)^{-1}(y_*) = \pi^{-1} \circ (\pi_{\text{eq}}^*)^{-1}(y_*) = \pi^{-1}(y_0)$$

is a singleton as $y_0 \in Y_0$. Thus we have the denseness of Y_1 in Y_{eq} , and then (X, T) is not thickly sensitive by Proposition 4.2. This finishes the proof. \square

REFERENCES

1. Ethan Akin, *Recurrence in topological dynamics*, The University Series in Mathematics, Plenum Press, New York, 1997, Furstenberg families and Ellis actions. MR 1467479 (2000c:37014)
2. Ethan Akin, Joseph Auslander, and Kenneth Berg, *When is a transitive map chaotic?*, Convergence in ergodic theory and probability (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ., vol. 5, de Gruyter, Berlin, 1996, pp. 25–40. MR 1412595 (97i:58106)
3. Ethan Akin and Eli Glasner, *Residual properties and almost equicontinuity*, J. Anal. Math. **84** (2001), 243–286. MR 1849204 (2002f:37020)
4. Ethan Akin and Sergii Kolyada, *Li-Yorke sensitivity*, Nonlinearity **16** (2003), no. 4, 1421–1433. MR 1986303 (2004c:37016)

5. Joseph Auslander, *Minimal flows and their extensions*, North-Holland Mathematics Studies, vol. 153, North-Holland Publishing Co., Amsterdam, 1988, Notas de Matemática [Mathematical Notes], 122. MR 956049 (89m:54050)
6. Joseph Auslander and James A. Yorke, *Interval maps, factors of maps, and chaos*, Tôhoku Math. J. (2) **32** (1980), no. 2, 177–188. MR 580273 (82b:58049)
7. François Blanchard, Bernard Host, and Alejandro Maass, *Topological complexity*, Ergodic Theory Dynam. Systems **20** (2000), no. 3, 641–662. MR 1764920 (2002b:37019)
8. Salomon Bochner, *Curvature and Betti numbers in real and complex vector bundles*, Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 225–253. MR 0084160 (18,819c)
9. ———, *A new approach to almost periodicity*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 2039–2043. MR 0145283 (26 #2816)
10. Tomasz Downarowicz, *Survey of odometers and Toeplitz flows*, Algebraic and topological dynamics, Contemp. Math., vol. 385, Amer. Math. Soc., Providence, RI, 2005, pp. 7–37. MR 2180227 (2006f:37009)
11. Robert Ellis and Walter H. Gottschalk, *Homomorphisms of transformation groups*, Trans. Amer. Math. Soc. **94** (1960), 258–271. MR 0123635 (23 #A960)
12. Harry Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, N.J., 1981, M. B. Porter Lectures. MR 603625 (82j:28010)
13. Eli Glasner, *A simple characterization of the set of μ -entropy pairs and applications*, Israel J. Math. **102** (1997), 13–27. MR 1489099 (98k:54076)
14. Eli Glasner and Benjamin Weiss, *Sensitive dependence on initial conditions*, Nonlinearity **6** (1993), no. 6, 1067–1075. MR 1251259 (94j:58109)
15. John Guckenheimer, *Sensitive dependence to initial conditions for one-dimensional maps*, Comm. Math. Phys. **70** (1979), no. 2, 133–160. MR 553966 (82c:58037)
16. Wen Huang, Danylo Khilko, Sergii Kolyada, and Guohua Zhang, *Dynamical compactness and sensitivity*, J. Differential Equations **260** (2016), no. 9, 6800–6827. MR 3461085
17. Wen Huang and Xiangdong Ye, *Devaney’s chaos or 2-scattering implies Li-Yorke’s chaos*, Topology Appl. **117** (2002), no. 3, 259–272. MR 1874089 (2003b:37017)
18. Sergii Kolyada and L’ubomír Snoha, *Some aspects of topological transitivity—a survey*, Iteration theory (ECIT 94) (Opava), Grazer Math. Ber., vol. 334, Karl-Franzens-Univ. Graz, Graz, 1997, pp. 3–35. MR 1644768
19. Sergii Kolyada, L’ubomír Snoha, and Sergeĭ Trofimchuk, *Noninvertible minimal maps*, Fund. Math. **168** (2001), no. 2, 141–163. MR 1852739 (2002j:37017)
20. Jian Li and Xiangdong Ye, *Recent development of chaos theory in topological dynamics*, Acta Math. Sin. (Engl. Ser.) **32** (2016), no. 1, 83–114. MR 3431162
21. Heng Liu, Li Liao, and Lidong Wang, *Thickly Syndetical Sensitivity of Topological Dynamical System*, Discrete Dyn. Nat. Soc. (2014), Art. ID 583431, 4. MR 3200824
22. T. K. Subrahmonian Moothathu, *Stronger forms of sensitivity for dynamical systems*, Nonlinearity **20** (2007), no. 9, 2115–2126. MR 2351026 (2008j:37014)
23. David Ruelle, *Dynamical systems with turbulent behavior*, Mathematical problems in theoretical physics (Proc. Internat. Conf., Univ. Rome, Rome, 1977), Lecture Notes in Phys., vol. 80, Springer, Berlin-New York, 1978, pp. 341–360. MR 518445 (81e:58032)
24. William A. Veech, *Almost automorphic functions on groups*, Amer. J. Math. **87** (1965), 719–751. MR 0187014 (32 #4469)
25. ———, *The equicontinuous structure relation for minimal Abelian transformation groups*, Amer. J. Math. **90** (1968), 723–732. MR 0232377 (38 #702)
26. ———, *Topological dynamics*, Bull. Amer. Math. Soc. **83** (1977), no. 5, 775–830. MR 0467705 (57 #7558)
27. Peter Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982. MR 648108 (84e:28017)
28. Xiangdong Ye and Tao Yu, *Sensitivity, proximal extension and higher order almost automorphy*, preprint, 2015.

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